Lecture 3
Continuous Random Variables

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Continuous random variables

Recall: The cumulative distribution function (CDF) of a random variable X is the function $F(x) : \mathbb{R} \to [0, 1]$ given by $F(x) = P(X \leqslant x)$

Definition 1: A random variable X is said to be of continuous type if its cumulative distribution function $F(x)$ is continuous on its support.

Definition 2: A random variable X is continuous if it takes infinite number of values within a interval or within the joint of several intervals on the real number.

Examples:

- length of time between machine failure;
- height of student in Lanzhou University;
- total biomass of sampling quadrants in grasslands;
- amount of precipitation received in a day.

Probability density function

We use probability mass function to represent the probability that a discrete random variable takes certain value. However, because continuous random variable can take infinite number of values, we cannot directly define a probability that a continuous random variable takes on a particular value.

Definition: Let $F(x)$ be the cumulative distribution function of a continuous random variable X. Probability density function, abbreviated as PDF, is a function $f(x)$ such that

$$F(x) = \int_{-\infty}^{x} f(s)ds$$

Probability density function

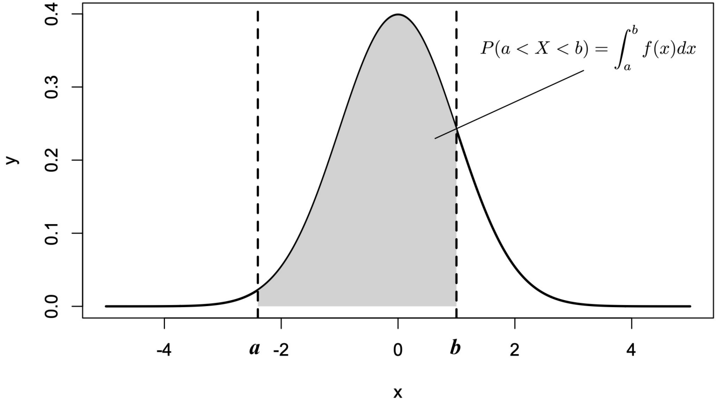
A probability density function of a continuous random variable X with space S that is an interval or union of intervals, is an integrable function $f(x)$ satisfying the following conditions:

- $f(x) \geqslant 0, \; x \in S$;
- $\int_S f(x)dx = 1$;
- If $(a, b) \subseteq S$, the probability of event $a < X < b$ is

$$P(a < X < b) = \int_a^b f(x)dx$$

Probability density function

Probability density function **is not** a probability. The area under the probability density function curve, i.e., the integration of probability density function, is probability.



Probability density function

Example: Let the random variable X denote the outcome when a point is selected at random from interval $[a, b]$. If the point is selected at random, the probability that the point is selected from the interval $[a, x]$ is $(x - a)/(b - a)$. Thus the cumulative distribution function of X is

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leqslant x \leqslant b, \\ 1, & b \leqslant x \end{cases}$$

which can be written as

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

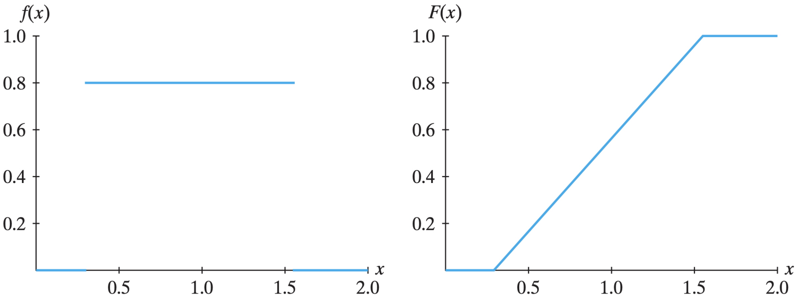
where

$$f(x) = \frac{1}{b - a}, \ a \leqslant x \leqslant b$$

Probability density function

The random variable X has a uniform distribution if its PDF is equal to a constant on its support. In particular, if the support is interval $[a, b]$, then

$$f(x) = \frac{1}{b-a}, \; a \leqslant x \leqslant b$$



PDF and CDF of a uniform distribution

Deriving probability density function

For discrete random variables, we can directly derive the probability mass function. For continuous random variables, however, we derive probability density function by taking the derivative of the cumulative distribution function:

$$f(x) = \frac{dF(x)}{dx}$$

Example: Suppose we select a point at random in the interior of a circle of radius 1. Let X be the distance of the selected point from the origin. What is the probability density function of X?

Answer: Because a point is selected at random, the event $X \leqslant x$ is equivant to the point lying in a circle of radius x. Hence, the CDF of X is

$$F(x) = P(X \leqslant x) = \pi x^2/\pi = x^2, \ 0 \leqslant x \leqslant 1$$

The PDF of X is

$$f(x) = \frac{dF(x)}{dx} = 2x, \ 0 \leqslant x \leqslant 1$$

Probability density function

Comments on probability density function:

- Probability mass function of a discrete random variable is bounded between 0 and 1. In contrast, probability density function does not have to be bounded. The restriction is that the area between PDF and the x-axis must equal 1;
- The PDF of a continuous random variable X does not need to be a continuous function. The cumulative distribution function must be continuous for a continuous random variable.

Mathematical expectation

Let $f(x)$ be the probability density function of continuous random variable X, the mathematical expectation of $u(x)$ is calculated as

$$E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx$$

The mean and variance of a continuous variable is calculated as

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx, \quad Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

The moment generating function of a continuous random variable is

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Mathematical expectation

Example: What is the mean and variance for $U(a, b)$?

$$
\begin{aligned}
E(X) &= \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b \\
&= \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= \int_a^b (x-\mu)^2 \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \frac{(x-\mu)^3}{3} \Big|_a^b \\
&= \frac{1}{b-a} \left[\frac{(b-a)^3}{24} - \frac{(a-b)^3}{24} \right] \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

Mathematical expectation

Example: X has a probability density function $f(x) = \lambda e^{-\lambda x},\ x \geqslant 0$. What is the expected value of X?

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x} dx$$

Integrating by parts with $u = \lambda x$ and $dv = e^{-\lambda x} dx$ so that $du = \lambda dx$ and $v = -\frac{1}{\lambda} e^{-\lambda x}$

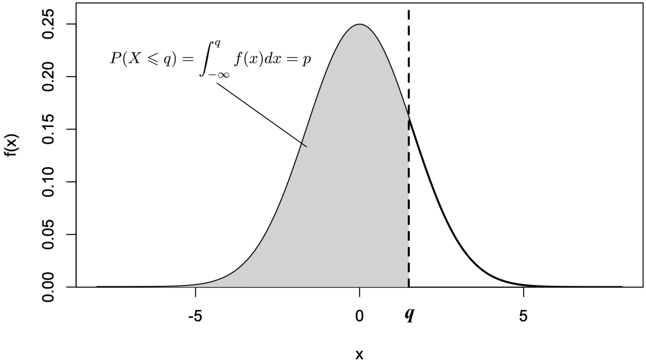
$$E(X) = (-xe^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x})\Big|_0^\infty = \frac{1}{\lambda}$$

Percentile

Definition: The 100*p*th quantile of a continuous random variable is a number q such that

$$\int_{-\infty}^{q} f(x)dx = p$$

The 50th percentile is called the **median** or the second quartile. The 25th and 75th percentiles are called the first and third **quartiles**, respectively.



Percentile

Example: The time X in months until the failure of a product as the PDF

$$f(x) = \frac{3x^2}{4^3} e^{-(x/4)^3}, \; x > 0$$

Its CDF is thus

$$F(x) = 1 - e^{-(x/4)^3}, \; x \geqslant 0$$

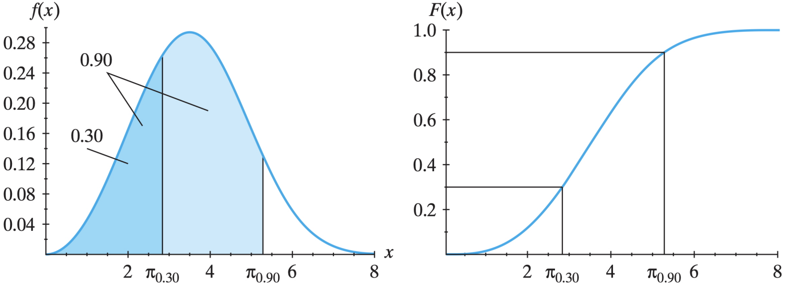
The 30th percentile $q_{0.3}$ is given by $F(q_{0.3}) = 0.3$. That is

$$\begin{aligned} 1 - e^{-(q_{0.3}/4)^3} &= 0.3 \\ \ln(0.7) &= -(q_{0.3}/4)^3 \\ q_{0.3} &= 2.84 \end{aligned}$$

Similarly, $q_{0.9}$ is found by $F(q_{0.9}) = 0.9$ so $q_{0.9} = 5.28$

Percentile

Example: The illustration of the 30th and 90th percentiles are shown in the graph of the PDF and CDF of the distribution for X:



Exponential distribution

A Poisson distribution models the number of occurrence in a given time interval. Not only is the number of occurrences a random variable; the waiting times between successive occurrences are also random variables, but is of a continuous type.

Let W denote the waiting time until the first occurrence during an observation of a Poisson process in which the mean number of occurrence in a unit time interval is λ. What is the distribution of W?

$$\begin{aligned} F(w) = P(W \leqslant w) &= 1 - P(W > w) \\ &= 1 - P(\text{no occurrence in } [0, w]) \\ &= 1 - \frac{(\lambda w)^0}{0!} e^{-\lambda w} = 1 - e^{-\lambda w} \end{aligned}$$

$$f(w) = F'(w) = \lambda e^{-\lambda w}$$

Exponential distribution

We often let $\lambda = 1/\theta$ and say that a random variable X has an exponential distribution if its PDF is defined as

$$f(x) = \frac{1}{\theta} e^{-x/\theta}$$

What is the meaning of the parameter θ?

The moment generating function of X is

$$\begin{aligned} M(t) &= \int_0^\infty e^{tx} \frac{1}{\theta} e^{-x/\theta} dx = \int_0^\infty \frac{1}{\theta} e^{(\theta t - 1)x/\theta} dx \\ &= \left[\frac{1}{\theta t - 1} e^{(\theta t - 1)x/\theta} \right] \Bigg|_0^\infty = \frac{1}{1 - \theta t}, \quad t < \frac{1}{\theta}. \end{aligned}$$

Thus, we have

$$M'(t) = \frac{\theta}{(1-\theta t)^2}, \quad M''(t) = \frac{2\theta^2}{(1-\theta t)^3}$$

$$\mu = M'(0) = \theta, \quad \sigma^2 = M''(0) - [M'(0)]^2 = \theta^2$$

Exponential distribution

From the derivation of exponential distribution, we see that its CDF is

$$F(x) = 1 - e^{-x/\theta}, \quad x \geqslant 0.$$

Note that for an exponential random variable X with mean θ, we have

$$\begin{aligned} P(X > x) = 1 - F(x) = 1 - (1 - e^{-x/\theta}) \\ = e^{-x/\theta}, \quad x \geqslant 0. \end{aligned}$$

Exponential distribution

Example: Let X be a random variable following exponential distribution with a mean of θ, what is the median of the distribution?

The CDF of X is

$$F(x) = 1 - e^{-x/\theta}$$

The median m is found by solving $F(m) = 0.5$. That is

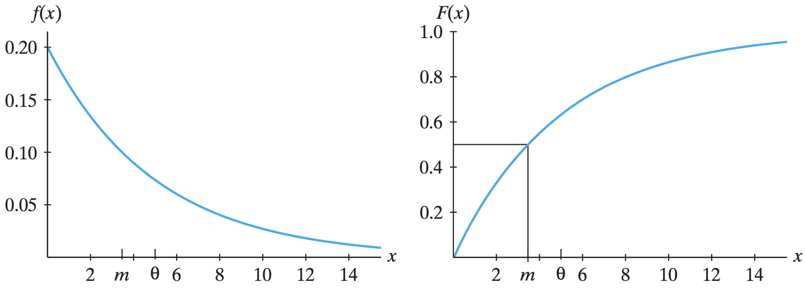
$$1 - e^{-m/\theta} = 0.5$$

Thus,

$$m = -\theta \ln(0.5) = \theta \ln(2)$$

Exponential distribution

For an exponential distribution, the median is typically smaller than the mean, i.e, $m = \theta \ln(2) < \theta$, as shown in the figure below.



Exponential distribution

Example: Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer?

Let X denote the waiting time in minutes until the first customer arrives and $\lambda = 1/3$ is the expected number of arrivals per minute. Thus

$$\theta = \frac{1}{3} = 3$$

and

$$f(x) = \frac{1}{3}e^{-x/3}, \quad x \geqslant 0$$

Hence,

$$P(X > 5) = \int_5^\infty \frac{1}{3}e^{-x/3}dx = e^{-5/3} = 0.1889$$

Exponential distribution

Theorem: Exponential distribution is **memoryless**, that is, if X has an exponential distribution, then

$$P(X > s + t | X > s) = P(X > t)$$

If X has an exponential distribution $f(x) = \frac{1}{\theta} e^{-x/\theta}$

$$P(X > t) = \int_t^{\infty} \frac{1}{\theta} e^{-x/\theta} dx = e^{-t/\theta}$$

$$\begin{aligned} P(X > s + t | X > s) &= \frac{P(X > s + t, X > s)}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} \\ &= \frac{e^{-(s+t)/\theta}}{e^{-s/\theta}} = e^{-t/\theta} \end{aligned}$$

Gamma distribution

In a Poisson process with mean λ, we now let W denote the waiting time until the αth occurrence. What is the distribution of W?

The CDF of W is given by

$$\begin{aligned} F(w) &= P(W \leqslant w) = 1 - P(W > w) \\ &= 1 - P(\textit{fewer than } \alpha \textit{ occurrences in } [0, w]) \\ &= 1 - \sum_{k=0}^{\alpha-1} \frac{(\lambda w)^k e^{-\lambda w}}{k!} \\ f(w) &= F'(w) = \lambda e^{-\lambda w} - e^{-\lambda w} \sum_{k=1}^{\alpha-1} \left[\frac{k(\lambda w)^{k-1}\lambda}{k!} - \frac{(\lambda w)^k \lambda}{k!} \right] \\ &= \lambda e^{-\lambda w} - e^{-\lambda w} \left[\lambda - \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} \right] \\ &= \frac{\lambda(\lambda w)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda w} \end{aligned}$$

Gamma distribution

The random variable W is said to have a **gamma distribution** if its PDF has this form. To generalize the PDF of W, we define the **gamma function**

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy, \quad t > 0$$

If $t > 1$, integrating the gamma function of t by parts yields

$$\begin{aligned}\Gamma(t) &= \Big[-y^{t-1}e^{-y}\Big]_0^\infty + \int_0^\infty (t-1) y^{t-2} e^{-y} dy \\ &= (t-1)\int_0^\infty y^{t-2} e^{-y} dy = (t-1)\Gamma(t-1)\end{aligned}$$

Whenever $t = n$, a positive integer, we have $\Gamma(n) = (n-1)\ldots(2)(1)\Gamma(1)$. However,

$$\Gamma(1) = \int_0^\infty e^{-y} dy = 1$$

We thus have $\Gamma(n) = (n-1)!$ for positive integers. For this reason, the gamma function is also called a generalized factorial.

Gamma distribution

The random variable X has a **gamma distribution** if its PDF is defined by

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}$$

The moment generating function of a gamma distribution is

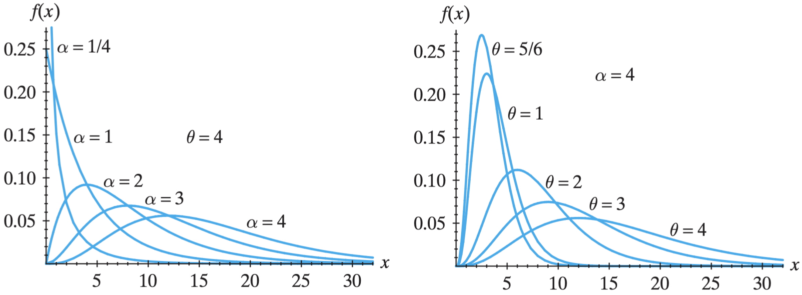
$$M(t) = \frac{1}{(1-\theta t)^\alpha}, \quad t < \frac{1}{\theta}$$

The mean and variance are

$$\mu = \alpha\theta \quad \textit{and} \quad \sigma^2 = \alpha\theta^2$$

Gamma distribution

A gamma PDF can take a variety of shape depending on the values of parameters α and θ.



Gamma distribution

Example: Suppose the number of customers per hour arriving at a shop follows a Poisson process with mean 30. That is, if a minute is our unit, then $\lambda = 1/2$. What is the probability that the shopkeeper will wait more than 5 minutes before both of the first two customers arrive?

If X denotes the waiting time in minutes until the second customer arrives, X has a gamma distribution with $\alpha = 2$ and $\theta = 1/\theta = 2$. Then,

$$\begin{aligned} P(X > 5) &= \int_5^\infty \frac{x^{2-1} e^{-x/2}}{\Gamma(2) 2^2} dx = \int_0^\infty \frac{x e^{-x/2}}{4} dx \\ &= \frac{1}{4} \left[(-2) x e^{-x/2} - 4 e^{-x/2} \right]_5^\infty \\ &= \frac{7}{2} e^{-5/2} = 0.287 \end{aligned}$$

Chi-square distribution

We now consider a special case of gamma distribution with $\theta = 2$ and $\alpha = r/2$. We say X has a **chi-square distribution** with r degrees of freedom, which we abbreviate by $\chi^2(r)$. Its PDF is:

$$f(x) = \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}, \quad x > 0$$

The moment generating function is

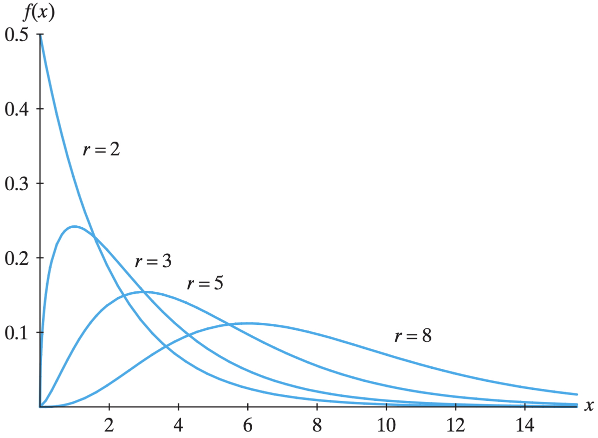
$$M(t) = (1 - 2t)^{-r/2}$$

The mean and variance are

$$\mu = r \quad \text{and} \quad \sigma^2 = 2r$$

Chi-square distribution

A chi-square distribution has only one parameter r, often called the degrees of freedom, that determines the shape of its PDF.



Chi-square distribution

Example: If X is $\chi^2(18)$, then the constant a such that $P(X > a) = 0.95$ is $a = 9.39$. Probabilities like this are important in statistical applications that we use special symbols for a.

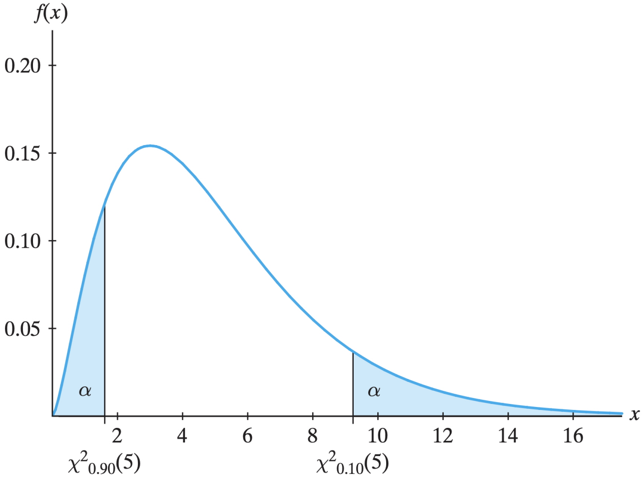
Let α be a positive probability and let X have a chi-square distribution with r degrees of freedom. Then $\chi^2_\alpha(r)$ is a number such that

$$P\left[X \geqslant \chi^2_\alpha(r)\right] = \alpha$$

That is, $\chi^2_\alpha(r)$ is the $100(1-\alpha)$th percentile of the chi-square distribution with r degrees of freedom.

Chi-square distribution

Graphically, $\chi^2_\alpha(r)$ is the upper 100αth percent point of the distribution.



Normal distribution

When observed over a large population, many variables have a “bell-shaped” relative frequency distribution, i.e., one that is approximately symmetric and relatively higher in the middle of the range of values than at the extremes.

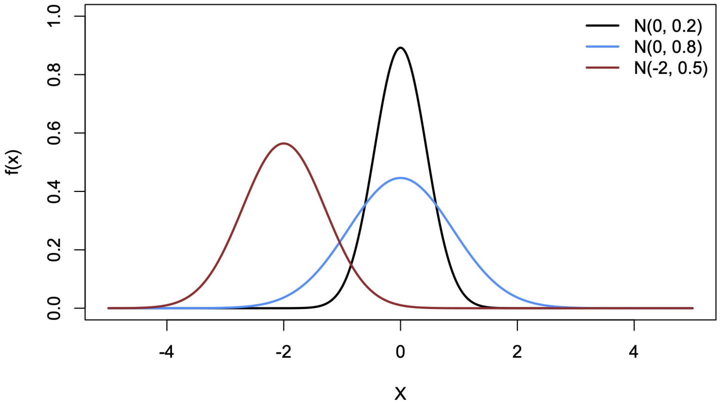
Examples include such variables as physical measurements (height, weight, length) of organisms, and repeated measurements of the same quantity on different occasions or by different observers. A very useful family of probability distributions for such variables are the **normal distributions.**

Normal distribution

A random variable X has a normal distribution if its PDF is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where μ and σ are parameters satisfying $-\infty < \mu < \infty$ and $0 < \sigma < \infty$.



Normal distribution

The moment generating function of a normal distribution is

$$\begin{aligned} M(t) &= \int_{-\infty}^{\infty} \frac{e^{tx}}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{(x-\mu)^2}{2\sigma^2}\Big] dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{1}{2\sigma^2}\big[x^2 - 2(\mu+\sigma^2 t)x + \mu^2\big]\Big] dx \end{aligned}$$

Rearrange the exponent:

$$x^2 - 2(\mu+\sigma^2 t)x + \mu^2 = [x - (\mu+\sigma^2 t)]^2 - 2\mu\sigma^2 t - \sigma^4 t^2$$

Hence,

$$\begin{aligned} M(t) &= \exp\Big(\frac{2\mu\sigma^2 t + \sigma^4 t^2}{2\sigma^2}\Big) \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{[x-(\mu+\sigma^2 t)^2]}{2\sigma^2}\Big] dx \\ &= \exp\Big(\mu t + \frac{\sigma^2 t^2}{2}\Big) \end{aligned}$$

Normal distribution

Based on the moment generating function, we can derive the mean and variance of a normal distribution:

$$M'(t) = (\mu + \sigma^2 t) \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

$$M''(t) = ((\mu + \sigma^2 t)^2 + \sigma^2) \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$$

Consequently,

$$E(X) = M'(0) = \mu$$

$$Var(X) = M''(0) - [M'(0)]^2 = \sigma^2$$

That is, the parameter μ and σ^2 in the PDF of the normally distributed X are the mean and variance of X. We often abbreviate normal distribution as $X \sim N(\mu, \sigma^2)$

Normal distribution

From the derivations above, we can see the mean and variance of a normal distribution if we are given the PDF or MGF.

Example: if the PDF of X is

$$f(x) = \frac{1}{\sqrt{32\pi}} \exp\left[- \frac{(x+7)^2}{32} \right]$$

then $X \sim N(-7, 16)$

Example: If the moment generating function of X is

$$M(t) = \exp(5t + 12t^2)$$

then $X \sim N(5, 24)$

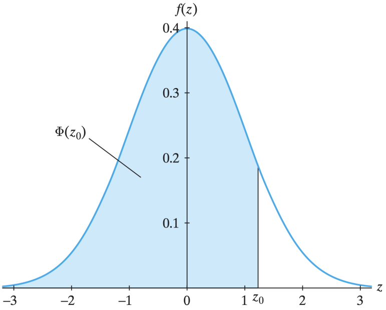
Normal distribution

A normal distribution with **mean 0 and standard deviation 1** is a **standard normal** distribution. Its probability density function is

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

and its cumulative distribution function is

$$\Phi(z) = P(x \leqslant z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$$



Normal distribution

Theorem: If $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X-\mu}{\sigma}$ is $N(0,1)$.

Proof: The cumulative distribution function of Z is

$$P(Z \leqslant z) = P(\frac{X-\mu}{\sigma} \leqslant z) = P(X \leqslant z\sigma + \mu)$$
$$= \int_{-\infty}^{z\sigma+\mu} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

We now use the change of variable integration given by $w = (x-\mu)/\sigma$ (i.e., $x = w\sigma + \mu$) to obtain

$$P(Z \leqslant z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw$$

This is useful because we can calculate probability of a normal distribution based on the probability of a standard normal distribution.

Calculate probability and quantile

Traditionally, probability and statistics course teach students using probability table to get probabilities. Now it is convenient to use software to directly calculate probabilities.

In R, there are functions you can use to get probability density, cumulative probability, or quantiles.

```
> dnorm(x = 1.5, mean = 0, sd = 1)
[1] 0.1295176
> pbinom(q = 3, size = 10, prob = 0.3)
[1] 0.6496107
> qchisq(p = 0.95, df = 1)
[1] 3.841459
```

Summary

Distribution	MGF	PDF	Mean	Variance
Exponential	$\frac{1}{1-\theta t}$	$\frac{1}{\theta} e^{-x/\theta}$	θ	θ^2
Gamma	$\frac{1}{(1-\theta t)^\alpha}$	$\frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-x/\theta}$	$\alpha\theta$	$\alpha\theta^2$
Chi-square	$\frac{1}{(1-2t)^{\frac{r}{2}}}$	$\frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}$	r	$2r$
Normal	$e^{\mu t+\frac{\sigma^2 t^2}{2}}$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2