Lecture 4
Multivariate Distributions

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September 29, 2025

Multivariate distribution

In many practical cases, it is possible, and often desirable, to take more than one measurement of a random observation. Moreover, we sometimes want to use these measurements to predict a third one. For example, we measure the GPA and extracurriculum activities of a student, and we give each of them a comprehensive evaluation score.

Definition: Let X and Y be two discrete random variables. Let S denote the two-dimensional space of X and Y. The probability that $X = x$ and $Y = y$ is denoted by $f(x, y) = P(X = x, Y = y)$. The function $f(x, y)$ is called the joint probability mass function.

Joint probability mass function

Example: Roll a pair of fair dice. For each of the 36 sampling points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. For example, if the outcome is $(3, 2)$, then the observed values are $X = 2,\ Y = 3$. What is the joint PMF of X and Y?

The event $X = 2,\ Y = 3$ can happen in one of two ways $(2, 3)$ or $(3, 2)$. So its probability is $2/36$. However, for event such as $X = 2,\ Y = 2$, it can only happen in one way. Thus, in general, the joint probability mass function is

$$f(x, y) = \begin{cases} \frac{1}{36} & x = y \\ \frac{1}{18} & x \neq y \end{cases}$$

Multinomial distribution

Suppose we have three mutually exclusive and exhaustive ways for an experiment to end: perfect, seconds, and defective. We repeat the experiment n independent times and the probability p_X, p_Y, $1 - p_X - p_Y$ of the three type of results. Let X and Y be the number of perfect and seconds. What is the joint probability mass function of X and Y?

The probability of having x perfects, y seconds, and $n - x - y$ defective is

$$p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

And it can be achieved in

$$\mathbf{C}_n^x \mathbf{C}_{n-x}^y = \frac{n!}{x!(n-x)!} \frac{(n-x)!}{y!(n-x-y)!} = \frac{n!}{x!y!(n-x-y)!}$$

Thus, the joint PMF is

$$f(x,y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

Marginal probability mass function

Let X and Y have the joint probability mass function $f(x, y)$ with space S. The probability mass function of X alone is called the marginal probability mass function of X and is defined by

$$f_X(x) = \sum_y f(x, y) \;\; x \in S_X$$

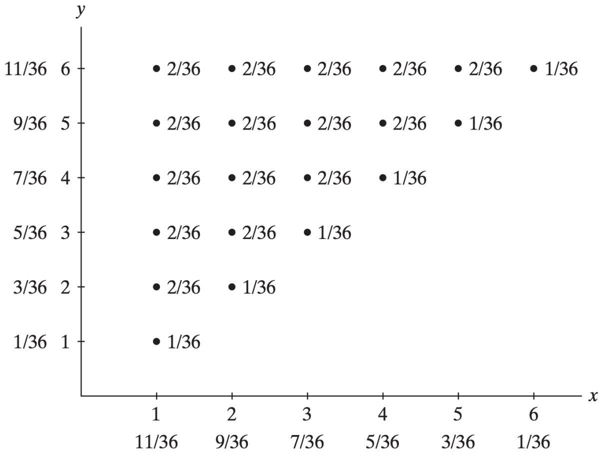
The random variables X and Y are independent if and only if, for every $x \in S_X$ and $y \in S_Y$,

$$f(x, y) = f_X(x) f_Y(y)$$

Otherwise, X ad Y are said to be dependent.

Marginal probability mass function

Example: In the dice rolling example mentioned above, what is the marginal probability mass function of X and Y? Are X and Y independent?



Marginal probability mass function

If X and Y has a multinomial distribution, are they independent?

It is easy to see by logic that X and Y both have a binomial distribution.

$$f_X(x) = \mathbf{C}_n^x p_X^x (1 - p_X)^{n-x}$$

$$f_Y(y) = \mathbf{C}_n^y p_Y^y (1 - p_Y)^{n-y}$$

Therefore,

$$f_X(x) f_Y(y) = \mathbf{C}_n^x \mathbf{C}_n^y p_X^x (1 - p_X)^{n-x} p_Y^y (1 - p_Y)^{n-y} \neq f(xy)$$

Thus, X and Y are not indepenent.

Mathematical expectation

Let X_1 and X_2 be random variables of the discrete type with the joint PMF $f(x_1, x_2)$ on the space S. If $u(X_1, X_2)$ is a function of these two random variables, then

$$E\left[u(X_1, X_2)\right] = \sum_{(x_1,x_2)\in S}\sum u(x_1, x_2) f(x_1, x_2)$$

if it exists, is called the mathematical expectation of $u(X_1, X_2)$.

If $u(X_1, X_2) = X_i$, then $E[u(X_1, X_2)] = E(X_i) = \mu_i$; if $u(X_1, X_2) = (X_i - \mu_i)^2$, then $E[u(X_1, X_2)] = E[(X_i - \mu_i)^2] = Var(X_i)$

Mathematical expectation

Example: There are eight chips in a bow: three marked $(0,0)$, two marked $(1,0)$, two marked $(0,1)$, and one marked $(1,1)$. A player selects a chip at random and is given the sum of the two coordinates in dollars as a prize. What is the expected prize money a play can get?

Let X_1 and X_2 denote the two coordinates. Their joint PMF is

$$f(x,y) = \frac{3 - x_1 - x_2}{8}, \quad x_1 = 0,1 \;\; and \;\; x_2 = 0,1$$

Thus,

$$\begin{aligned} E(X_1 + X_2) &= \sum_{x_2=0}^{1} \sum_{x_1=0}^{1} (x_1 + x_2) \frac{3 - x_1 - x_2}{8} \\ &= (0)(\frac{3}{8}) + (1)(\frac{2}{8}) + (1)(\frac{2}{8}) + (2)(\frac{1}{8}) = \frac{3}{4} \end{aligned}$$

Correlation coefficient

Let $u(X, Y) = (X - \mu_X)(Y - \mu_Y)$, then

$$E[u(X, Y)] = E[(X - \mu_X)(Y - \mu_Y)] = Cov(X, Y) = \sigma_{XY}$$

is called the covariance of X and Y.

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

is called the correlation coefficient of X and Y.

A commonly used formula to calculate covariance:

$$\begin{aligned} Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y) \\ &= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y \\ &= E(XY) - \mu_X \mu_Y \end{aligned}$$

Correlation coefficient

Example: Let X and Y have the joint PMF

$$f(x,y) = \frac{x+2y}{18}, \quad x = 1,2 \;\; and \;\; y = 1,2$$

What is the correlation coefficient of X and Y?

The marginal PMF are respectively

$$f_X(x) = \sum_{y=1}^{2} \frac{x+2y}{18} = \frac{x+3}{9}$$

$$f_Y(y) = \sum_{x=1}^{2} \frac{x+2y}{18} = \frac{3+4y}{18}$$

The mean and variance of X are

$$\mu_X = \sum_{x=1}^{2} x\frac{x+3}{9} = (1)\frac{4}{9} + (2)\frac{5}{9} = \frac{14}{9}$$

Correlation coefficient

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \sum_{x=1}^{2} x^2 \frac{x+3}{9} - \left(\frac{14}{9}\right)^2 = \frac{20}{81}$$

Similarly, we get the mean and variance of Y

$$\mu_Y = \frac{29}{18} \quad \sigma_Y^2 = \frac{77}{324}$$

The covariance of X and Y

$$\begin{aligned} Cov(X,Y) &= \sum_{x=1}^{2}\sum_{y=1}^{2} xy\frac{x+2y}{18} - \frac{14}{9}\frac{29}{18} \\ &= (1)(1)\frac{3}{18} + (2)(1)\frac{4}{18} + (1)(2)\frac{5}{18} + (2)(2)\frac{6}{18} - \frac{14}{9}\frac{29}{18} \\ &= -\frac{1}{162} \end{aligned}$$

$$\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = -0.025$$

Correlation coefficient

Proposition: If X and Y are independent, $Cov(X, Y) = 0$.

$$\begin{aligned} E(XY) &= \sum_{S_X} \sum_{S_Y} xyf(x, y) \\ &= \sum_{S_X} \sum_{S_Y} xyf_X(x)f_Y(y) \\ &= \sum_{S_X} xf_X(x) \sum_{S_Y} yf_Y(y) \\ &= \mu_X \mu_Y \end{aligned}$$

Thus, we have

$$Cov(X, Y) = E(XY) - \mu_X \mu_Y = 0$$

Correlation coefficient

If $Cov(X, Y) = 0$, are X and Y necessarily independent?

Example: Let X and Y have the joint PMF

$$f(x, y) = \frac{1}{3}, \quad (x, y) = (0, 1), (1, 0), (2, 1).$$

It is easy to get the marginal PMF of X and Y:

$$f_X(x) = \frac{1}{3},\ x = 0, 1, 2; \quad f_Y(y) = \begin{cases} \frac{1}{3}, & y = 0 \\ \frac{2}{3}, & y = 1 \end{cases}$$

Thus, $\mu_X = 1$ amd $\mu_Y = 2/3$. Then

$$\begin{aligned} Cov(X, Y) &= E(XY) - \mu_X \mu_Y \\ &= (0)(1)\frac{1}{3} + (1)(0)\frac{1}{3} + (2)(1)\frac{1}{3} - (1)\frac{2}{3} \\ &= 0 \end{aligned}$$

It is obvious that $f(x, y) \neq f_X(x) f_Y(y)$. Thus, X and Y are dependent.

Conditional distributions

Let X and Y have a joint discrete distribution with PMF $f(x, y)$ on space S. Say the marginal PMF are $f_X(x)$ and $f_Y(y)$ respectively. Let event $A = \{X = x\}$ and event $B = \{Y = y\}$. Thus $A \cap B = \{X = x, Y = y\}$. Because $P(A \cap B) = P(X = x, Y = y) = f(x, y)$ and $P(B) = P(Y = y) = f_Y(y)$, the conditional probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{f(x, y)}{f_Y(y)}$$

Definition: The conditional probability mass function of X, given that $Y = y$, is defined by

$$g(x|y) = \frac{f(x, y)}{f_Y(y)}$$

provided that $f_Y(y) > 0$

Conditional distributions

Example: Let X and Y have the joint PMF

$$f(x,y) = \frac{x+y}{21}, \quad x = 1,2,3, \quad y = 1,2.$$

Find the conditional distribution $g(x|y)$.

We first calculate marginal PMF of y:

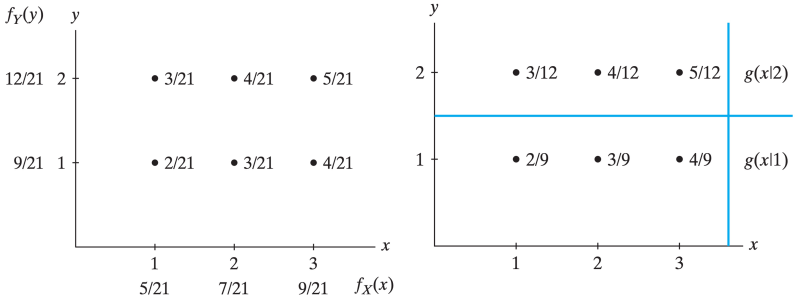
$$f_Y(y) = \sum_{x=1}^{3} \frac{x+y}{21} = \frac{y+2}{7}, \quad y = 1,2$$

Thus, the conditional PMF of X given Y is

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{(x+y)/21}{(y+2)/7} = \frac{x+y}{3y+6}$$

Conditional distribution

Similar to conditional probability, we can visualize the joint, marginal, and conditional PMF.



(Graphic illustration of joint, marginal and conditional PMF.)

Conditional expectation

Because conditional PMF is a PMF, we thus can define conditional expectation the same way we define mathematical expectation:

$$E\left[u(Y)|X=x\right]=\sum_{y}u(y)g(y|x)$$

Conditional mean and conditional variance are defined by

$$\mu_{Y|X}=E(Y|X)=\sum_{y}yg(y|x)$$

$$\sigma^2_{Y|X}=E\left[(Y-\mu_{Y|X})^2|X\right]=\sum_{y}(y-\mu_{Y|X})^2g(y|x)$$

Conditional expectation

Example: Let X and Y have a multinomial PMF with parameters n, p_X, and p_Y. That is,

$$f(x,y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_x - p_Y)^{n-x-y}$$

What is the conditional mean of X given Y?

We know that the marginal distribution of Y is binomial,. i.e.,

$$f_Y(y) = \frac{n!}{y!(n-y)!} p_Y^y (1 - p_Y)^{n-y}$$

Thus, the conditional PMF of X given Y is

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{(n-y)!}{x!(n-y-x)!} (\frac{p_X}{1-p_Y})^x (1 - \frac{p_X}{1-p_Y})^{n-y-x}$$

This is a binomial distribution with parameters $n-y$ and $\frac{p_X}{1-p_Y}$. Thus, the conditional mean is $(n-y)\frac{p_X}{1-p_Y}$.

Multivariate distribution of continuous random variables

The idea of joint distributions of discrete random variables can be extended to that of continuous random variables. The **joint probability density function** of two continuous random variables is an integrable function $f(x, y)$ such that

- $f(x, y) \geqslant 0$, where $f(x, y) = 0$ when (x, y) is not in the space of X and Y;
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$;
- $P(X, Y) \in A = \int \int_{A} f(x, y) dxdy$

Multivariate distribution of continuous random variables

The **marginal probability density** function of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy, \quad x \in S_X;$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)dx, \quad y \in S_Y;$$

X and Y are **independent** if and only if $f(x, y) = f_X(x)f_Y(y)$

Multivariate distribution of continuous random variables

The correlation coefficient of two continuous random variables X and Y is defined in the same way as the discrete random variables as

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

The **Conditional probability density function** of X, given that $Y = y$, is

$$f(x|y) = \frac{f(x, y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

Multivariate distribution of continuous random variables

Example: Let X and Y have the joint PDF

$$f(x, y) = 1, \quad x \leqslant y \leqslant x + 1, \quad 0 \leqslant x \leqslant 1.$$

Find the marginal PDF and the correlation coefficient of X and Y.

The marginal PDFs of X and Y are

$$f_X(x) = \int_x^{x+1} 1 dy = 1, \quad 0 \leqslant x \leqslant 1$$

$$f_Y(y) = \begin{cases} \displaystyle\int_0^y 1 dx = y, \quad 0 \leqslant y \leqslant 1, \\ \displaystyle\int_{y-1}^1 1 dx = 2 - y, \quad 1 \leqslant y \leqslant 2. \end{cases}$$

Multivariate distribution of continuous random variables

The mean and variance of X and Y are

$$\mu_X = \int_0^1 x \cdot 1 dx = \frac{1}{2}$$

$$\mu_Y = \int_0^1 y \cdot y dy + \int_1^2 y \cdot (2-y) dy = \frac{1}{3} + \frac{2}{3} = 1$$

$$E(X^2) = \int_0^1 x^2 \cdot 1 dx = \frac{1}{3}$$

$$E(Y^2) = \int_0^1 y^2 \cdot y dy + \int_1^2 y^2 \cdot (2-y) dy = \frac{7}{6}$$

$$E(XY) = \int_0^1 \int_x^{x+1} xy \cdot 1 dy dx = \int_0^1 \frac{1}{2} x(2x+1) dx = \frac{7}{12}$$

Multivariate distribution of continuous random variables

$$\sigma_X^2 = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12}$$

$$\sigma_Y^2 = \frac{7}{6} - 1^2 = \frac{1}{6}$$

$$\sigma_{XY} = \frac{7}{12} - \left(\frac{1}{2}\right)(1) = \frac{1}{12}$$

Therefore, the correlation coefficient is

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{1/12}{\sqrt{(1/12)(1/6)}} = \frac{\sqrt{2}}{2}$$

Multivariate normal distribution

A very commonly used multivariate distribution is the multivariate normal distribution. Random variables X and Y have a bivariate normal distribution if its joint PDF is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x,y)}{2}\right],$$

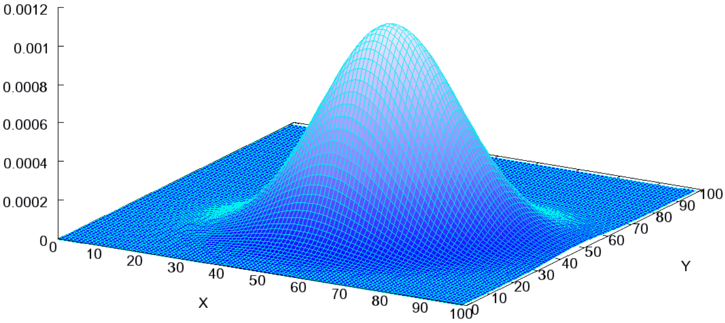
where

$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

Here, μ_X and μ_Y are the mean of X and Y, σ_X and σ_Y are the standard deviation of X and Y, and ρ is the correlation coefficient.

Multivariate normal distribution

A bivariate normal distribution has a typical PDF figure as follows.



Multivariate normal distribution

If random variables X and Y have a bivariate normal distribution, then the marginal distribution of X and Y are both normal.

$$
\begin{aligned}
q(x,y) &= \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] \\
&= \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X} - \rho\frac{y-\mu_Y}{\sigma_Y}\right)^2 + (1-\rho^2)\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right] \\
&= \frac{1}{\sigma_X^2(1-\rho^2)}\left(x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2
\end{aligned}
$$

Multivariate normal distribution

Thus, the marginal distribution of Y is

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f(x,y)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x,y)}{2}\right]dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right] \\
&\quad \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\sigma_X^2(1-\rho^2)}\left(x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)^2\right]dx \\
&= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right](\sigma_X\sqrt{2\pi}\sqrt{1-\rho^2}) \\
&= \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]
\end{aligned}
$$

Thus, the marginal distribution of Y is $N(\mu_Y, \sigma_Y^2)$. Using the procedure, it is obvious that $X \sim N(\mu_X, \sigma_X^2)$.

Multivariate normal distribution

If If random variables X and Y have a bivariate normal distribution, then the conditional distribution of X given Y is normal.

The joint PDF is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{q(x,y)}{2}\right],$$

where

$$q(x,y) = \frac{1}{\sigma_X^2(1-\rho^2)}\left(x-\mu_X-\rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y)\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2$$

The marginal PDF of Y is

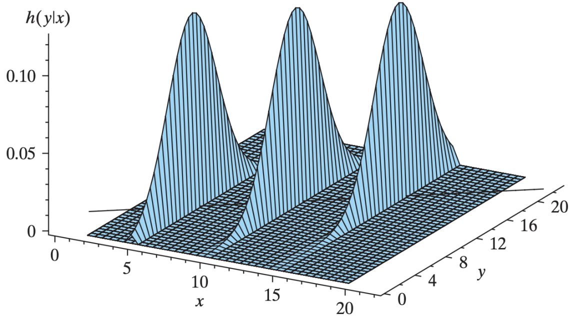
$$f_Y(y) = \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right]$$

Multivariate normal distribution

The conditional distribution of X given Y is thus

$$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{1}{\sigma_X\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left[-\frac{[x-\mu_X-\rho(\sigma_X/\sigma_Y)(y-\mu_Y)]^2}{2\sigma_X^2(1-\rho^2)}\right]$$

Thus, $g(x|y)$ is $N\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y), (1-\rho^2)\sigma_X^2\right)$.



(Illustration of conditional distribution of a bivariate normal distribution)

Multivariate normal distribution

We can extend the case of bivariate normal distribution to more than two variables. For more k variables, we write the PDF of a multivariate normal distribution in matrix notation:

$$f(\boldsymbol{x}) = \frac{1}{\sqrt{(2\pi)^k |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2} (\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{x} - \boldsymbol{\mu}) \right)$$

where $\boldsymbol{x}$ and $\boldsymbol{\mu}$ are column vectors of the variables and its means, $\boldsymbol{\Sigma}$ is the $k \times k$ variance covariance matrix, i.e.,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}; \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{bmatrix}; \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2_{X_1} & \sigma_{X_1 X_2} & \cdots & \sigma_{X_1 X_k} \\ \sigma_{X_1 X_1} & \sigma^2_{X_2} & \cdots & \sigma_{X_2 X_k} \\ \vdots & \vdots & & \vdots \\ \sigma_{X_k X_1} & \sigma^2_{X_k X_2} & \cdots & \sigma^2_{X_k} \end{bmatrix}$$