Lecture 5
Continuous Random Variables

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Continuous random variables

Recall: The cumulative distribution function (CDF) of a random variable X is the function $F(x) : \mathbb{R} \to [0, 1]$ given by $F(x) = P(X \leqslant x)$

Definition 1: A random variable X is said to be of continuous type if its cumulative distribution function $F(x)$ is continuous on its support.

Definition 2: A random variable X is continuous if it takes infinite number of values within a interval or within the joint of several intervals on the real number.

Examples:

- length of time between machine failure;
- height of student in Lanzhou University;
- total biomass of sampling quadrants in grasslands;
- amount of precipitation received in a day.

Probability density function

We use probability mass function to represent the probability that a discrete random variable takes certain value. However, because continuous random variable can take infinite number of values, we cannot directly define a probability that a continuous random variable takes on a particular value.

Definition: Let $F(x)$ be the cumulative distribution function of a continuous random variable X. Probability density function, abbreviated as PDF, is a function $f(x)$ such that

$$F(x) = \int_{-\infty}^{x} f(s)ds$$

Probability density function

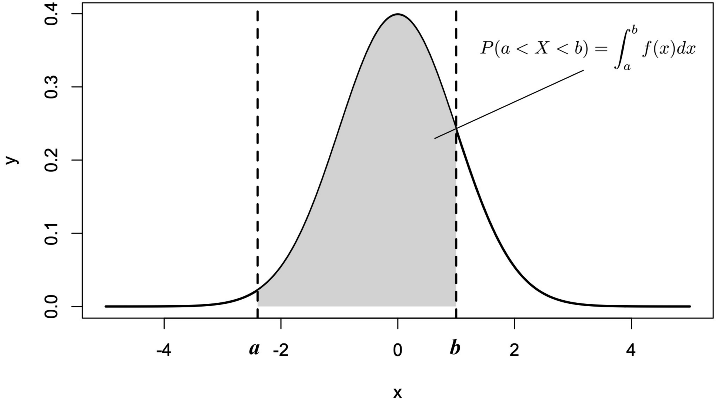
A probability density function of a continuous random variable X with space S that is an interval or union of intervals, is an integrable function $f(x)$ satisfying the following conditions:

- $f(x) \geqslant 0, \ x \in S$;
- $\int_S f(x)dx = 1$;
- If $(a, b) \subseteq S$, the probability of event $a < X < b$ is

$$P(a < X < b) = \int_a^b f(x)dx$$

Probability density function

Probability density function **is not** a probability. The area under the probability density function curve, i.e., the integration of probability density function, is probability.



Probability density function

Example: Let the random variable X denote the outcome when a point is selected at random from interval $[a, b]$. If the point is selected at random, the probability that the point is selected from the interval $[a, x]$ is $(x - a)/(b - a)$. Thus the cumulative distribution function of X is

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leqslant x \leqslant b, \\ 1, & b \leqslant x \end{cases}$$

which can be written as

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

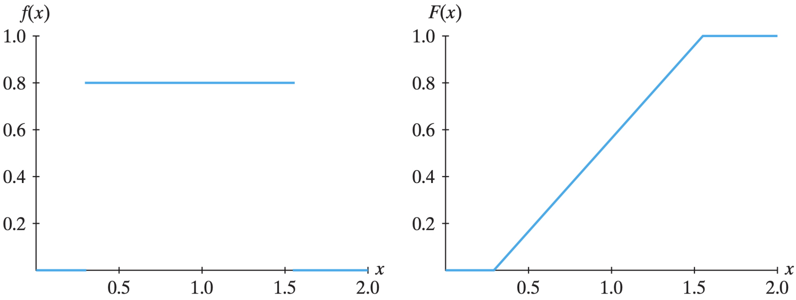
where

$$f(x) = \frac{1}{b-a}, \ a \leqslant x \leqslant b$$

Probability density function

The random variable X has a uniform distribution if its PDF is equal to a constant on its support. In particular, if the support is interval $[a, b]$, then

$$f(x) = \frac{1}{b-a}, \ a \leqslant x \leqslant b$$



PDF and CDF of a uniform distribution

Deriving probability density function

For discrete random variables, we can directly derive the probability mass function. For continuous random variables, however, we derive probability density function by taking the derivative of the cumulative distribution function:

$$f(x) = \frac{dF(x)}{dx}$$

Example: Suppose we select a point at random in the interior of a circle of radius 1. Let X be the distance of the selected point from the origin. What is the probability density function of X?

Answer: Because a point is selected at random, the event $X \leqslant x$ is equivant to the point lying in a circle of radius x. Hence, the CDF of X is

$$F(x) = P(X \leqslant x) = \pi x^2/\pi = x^2,\ 0 \leqslant x \leqslant 1$$

The PDF of X is

$$f(x) = \frac{dF(x)}{dx} = 2x,\ 0 \leqslant x \leqslant 1$$

Probability density function

Comments on probability density function:

- Probability mass function of a discrete random variable is bounded between 0 and 1. In contrast, probability density function does not have to be bounded. The restriction is that the area between PDF and the x-axis must equal 1;
- The PDF of a continuous random variable X does not need to be a continuous function. The cumulative distribution function must be continuous for a continuous random variable.

Mathematical expectation

Let $f(x)$ be the probability density function of continuous random variable X, the mathematical expectation of $u(x)$ is calculated as

$$E[u(x)] = \int_{-\infty}^{\infty} u(x)f(x)dx$$

The mean and variance of a continuous variable is calculated as

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx, \quad Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

The moment generating function of a continuous random variable is

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Mathematical expectation

Example: What is the mean and variance for $U(a, b)$?

$$
\begin{aligned}
E(X) &= \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a} \frac{x^2}{2} \Big|_a^b \\
&= \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{a+b}{2}
\end{aligned}
$$

$$
\begin{aligned}
Var(X) &= \int_a^b (x-\mu)^2 \frac{1}{b-a} dx \\
&= \frac{1}{b-a} \frac{(x-\mu)^3}{3} \Big|_a^b \\
&= \frac{1}{b-a} \left[\frac{(b-a)^3}{24} - \frac{(a-b)^3}{24} \right] \\
&= \frac{(b-a)^2}{12}
\end{aligned}
$$

Mathematical expectation

Example: X has a probability density function $f(x) = \lambda e^{-\lambda x},\ x \leqslant 0$. What is the expected value of X?

$$E(X) = \int_0^\infty x\lambda e^{-\lambda x} dx$$

Integrating by parts with $u = \lambda x$ and $dv = e^{-\lambda x} dx$ so that $du = \lambda dx$ and $v = -\frac{1}{\lambda} e^{-\lambda x}$

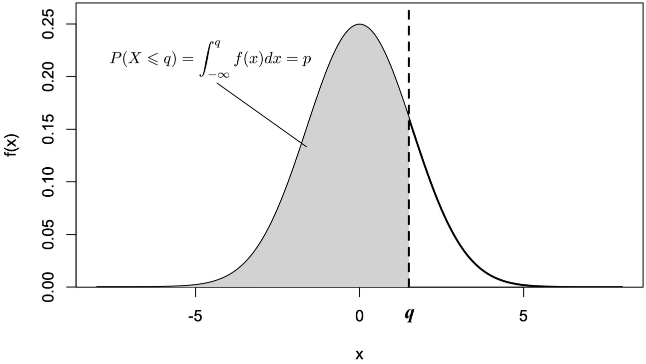
$$E(X) = (-xe^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x})\Big|_0^\infty = \frac{1}{\lambda}$$

Percentile

Definition: The 100pth quantile of a continuous random variable is a number q such that

$$\int_{-\infty}^{q} f(x)dx = p$$

The 50th percentile is called the **median** or the second quartile. The 25th and 75th percentiles are called the first and third **quartiles**, respectively.



Percentile

Example: The time X in months until the failure of a product as the PDF

$$f(x) = \frac{3x^2}{4^3} e^{-(x/4)^3}, \ x > 0$$

Its CDF is thus

$$F(x) = 1 - e^{-(x/4)^3}, \ x \geqslant 0$$

The 30th percentile $q_{0.3}$ is given by $F(q_{0.3}) = 0.3$. That is

$$\begin{aligned} 1 - e^{-(q_{0.3}/4)^3} &= 0.3 \\ \ln(0.7) &= -(q_{0.3}/4)^3 \\ q_{0.3} &= 2.84 \end{aligned}$$

Similarly, $q_{0.9}$ is found by $F(q_{0.9}) = 0.9$ so $q_{0.9} = 5.28$

Percentile

Example: The illustration of the 30th and 90th percentiles are shown in the graph of the PDF and CDF of the distribution for X: