Lecture 7
Multivariate Distributions

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Multivariate distribution

In many practical cases, it is possible, and often desirable, to take more than one measurement of a random observation. Moreover, we sometimes want to use these measurements to predict a third one. For example, we measure the GPA and extracurriculum activities of a student, and we give each of them a comprehensive evaluation score.

Definition: Let X and Y be two discrete random variables. Let S denote the two-dimensional space of X and Y. The probability that $X = x$ and $Y = y$ is denoted by $f(x, y) = P(X = x, Y = y)$. The function $f(x, y)$ is called the joint probability mass function.

Joint probability mass function

Example: Roll a pair of fair dice. For each of the 36 sampling points with probability $1/36$, let X denote the smaller and Y the larger outcome on the dice. For example, if the outcome is $(3, 2)$, then the observed values are $X = 2,\ Y = 3$. What is the joint PMF of X and Y?

The event $X = 3,\ Y = 3$ can happen in one of two ways $(2, 3)$ or $(3, 2)$. So its probability is $2/36$. However, for event such as $X = 2,\ Y = 2$, it can only happen in one way. Thus, in general, the joint probability mass function is

$$f(x, y) = \begin{cases} \frac{1}{36} & x = y \\ \frac{1}{18} & x \neq y \end{cases}$$

Multinomial distribution

Suppose we have three mutually exclusive and exhaustive ways for an experiment to end: perfect, seconds, and defective. We repeat the experiment n independent times and the probability p_X, p_Y, $1 - p_X - p_Y$ of the three type of results. Let X and Y be the number of perfect and seconds. What is the joint probability mass function of X and Y?

The probability of having x perfects, y seconds, and $n - x - y$ defective is

$$p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

And it can be achieved in

$$\mathbf{C}_n^x \mathbf{C}_{n-x}^y = \frac{n!}{x!(n-x)!} \frac{(n-x)!}{y!(n-x-y)!} = \frac{n!}{x!y!(n-x-y)!}$$

Thus, the joint PMF is

$$f(x, y) = \frac{n!}{x!y!(n-x-y)!} p_X^x p_Y^y (1 - p_X - p_Y)^{n-x-y}$$

Marginal probability mass function

Let X and Y have the joint probability function $f(x, y)$ with space S. The probability mass function of X alone is called the marginal probability mass function of X and is defined by

$$f_X(x) = \sum_y f(x, y) \quad x \in S_X$$

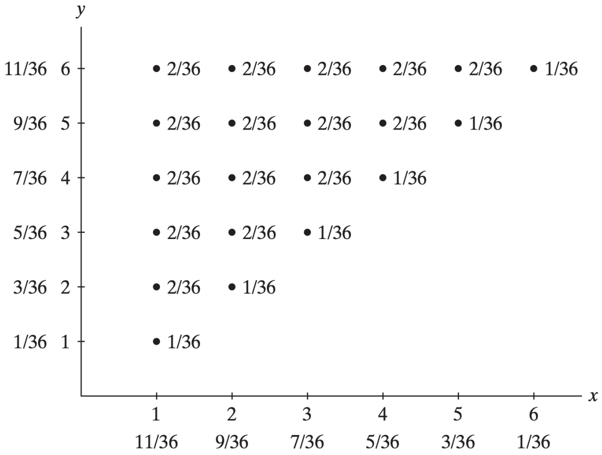
The random variables X and Y are independent if and only if, for every $x \in S_X$ and $y \in S_Y$,

$$f(x, y) = f_X(x) f_Y(y)$$

Otherwise, X ad Y are said to be dependent.

Marginal probability mass function

Example: In the dice rolling example mentioned above, what is the marginal probability mass function of X and Y? Are X and Y independent?



Marginal probability mass function

If X and Y has a multinomial distribution, are they independent?

It is easy to see by logic that X and Y both have a binomial distribution.

$$f_X(x) = \mathbf{C}_n^x p_X^x (1 - p_X)^{n-x}$$

$$f_Y(y) = \mathbf{C}_n^y p_Y^y (1 - p_Y)^{n-y}$$

Therefore,

$$f_X(x) f_Y(y) = \mathbf{C}_n^x \mathbf{C}_n^y p_X^x (1 - p_X)^{n-x} p_Y^y (1 - p_Y)^{n-y} \neq f(xy)$$

Thus, X and Y are not indepenent.

Mathematical expectation

Let X_1 and X_2 be random variables of the discrete type with the joint PMF $f(x_1, x_2)$ on the space S. If $u(X_1, X_2)$ is a function of these two random variables, then

$$E\left[u(X_1, X_2)\right] = \sum\sum_{(x_1, x_2) \in S} u(x_1, x_2) f(x_1, x_2)$$

if it exists, is called the mathematical expectation of $u(X_1, X_2)$.

If $u(X_1, X_2) = X_i$, then $E[u(X_1, X_2)] = E(X_i) = \mu_i$; if $u(X_1, X_2) = (X_i - \mu_i)^2$, then $E[u(X_1, X_2)] = E[(X_i - \mu_i)^2] = Var(X_i)$

Mathematical expectation

Example: There are eight chips in a bow: three marked $(0, 0)$, two marked $(1, 0)$, two marked $(0, 1)$, and one marked $(1, 1)$. A player selects a chip at random and is given the sum of the two coordinates in dollars as a prize. What is the expected prize money a play can get?

Let X_1 and X_2 denote the two coordinates. Their joint PMF is

$$f(x, y) = \frac{3 - x_1 - x_2}{8}, \quad x_1 = 0, 1 \;\; \textit{and} \;\; x_2 = 0, 1$$

Thus,

$$E(X_1 + X_2) = \sum_{x_2=0}^{1} \sum_{x_1=0}^{1} (x_1 + x_2) \frac{3 - x_1 - x_2}{8}$$
$$= (0)(\frac{3}{8}) + (1)(\frac{2}{8}) + (1)(\frac{2}{8}) + (2)(\frac{1}{8}) = \frac{3}{4}$$

Correlation coefficient

Let $u(X, Y) = (X - \mu_X)(Y - \mu_Y)$, then

$$E[u(X, Y)] = E[(X - \mu_X)(Y - \mu_Y)] = Cov(X, Y) = \sigma_{XY}$$

is called the covariance of X and Y.

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

is called the correlation coefficient of X and Y.

A commonly used formula to calculate covariance:

$$\begin{aligned} Cov(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E(XY - \mu_X Y - \mu_Y X + \mu_X \mu_Y) \\ &= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X \mu_Y \\ &= E(XY) - \mu_X \mu_Y \end{aligned}$$

Correlation coefficient

Example: Let X and Y have the joint PMF

$$f(x,y) = \frac{x+2y}{18}, \quad x = 1,2 \;\; and \;\; y = 1,2$$

What is the correlation coefficient of X and Y?

The marginal PMF are respectively

$$f_X(x) = \sum_{y=1}^{2} \frac{x+2y}{18} = \frac{x+3}{9}$$

$$f_Y(y) = \sum_{x=1}^{2} \frac{x+2y}{18} = \frac{3+4y}{18}$$

The mean and variance of X are

$$\mu_X = \sum_{x=1}^{2} x\frac{x+3}{9} = (1)\frac{4}{9} + (2)\frac{5}{9} = \frac{14}{9}$$

Correlation coefficient

$$\sigma_X^2 = E(X^2) - \mu_X^2 = \sum_{x=1}^{2} x^2 \frac{x+3}{9} - \left(\frac{14}{9}\right)^2 = \frac{20}{81}$$

Similarly, we get the mean and variance of Y

$$\mu_Y = \frac{29}{18} \quad \sigma_Y^2 = \frac{77}{324}$$

The covariance of X and Y

$$\begin{aligned} Cov(X, Y) &= \sum_{x=1}^{2} \sum_{y=1}^{2} xy \frac{x+2y}{18} - \frac{14}{9} \frac{29}{18} \\ &= (1)(1)\frac{3}{18} + (2)(1)\frac{4}{18} + (1)(2)\frac{5}{18} + (2)(2)\frac{6}{18} - \frac{14}{9}\frac{29}{18} \\ &= -\frac{1}{162} \end{aligned}$$

$$\rho = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = -0.025$$

Correlation coefficient

Proposition: If X and Y are independent, $Cov(X, Y) = 0$.

$$\begin{aligned} E(XY) &= \sum_{S_X} \sum_{S_Y} xyf(x, y) \\ &= \sum_{S_X} \sum_{S_Y} xyf_X(x)f_Y(y) \\ &= \sum_{S_X} xf_X(x) \sum_{S_Y} yf_Y(y) \\ &= \mu_X \mu_Y \end{aligned}$$

Thus, we have

$$Cov(X, Y) = E(XY) - \mu_X \mu_Y = 0$$

Correlation coefficient

If $Cov(X, Y) = 0$, are X and Y necessarily independent?

Example: Let X and Y have the joint PMF

$$f(x, y) = \frac{1}{3}, \quad (x, y) = (0, 1), (1, 0), (2, 1).$$

It is easy to get the marginal PMF of X and Y:

$$f_X(x) = \frac{1}{3},\ x = 0, 1, 2; \quad f_Y(y) = \begin{cases} \frac{1}{3},\ y = 0 \\ \frac{2}{3},\ y = 1 \end{cases}$$

Thus, $\mu_X = 1$ amd $\mu_Y = 2/3$. Then

$$\begin{aligned} Cov(X, Y) &= E(XY) - \mu_X \mu_Y \\ &= (0)(1)\frac{1}{3} + (1)(0)\frac{1}{3} + (2)(1)\frac{1}{3} - (1)\frac{2}{3} \\ &= 0 \end{aligned}$$

It is obvious that $f(x, y) \neq f_X(x) f_Y(y)$. Thus, X and Y are dependent.