Lecture 4
Linear Mixed–Effects Model

Chao Song

College of Ecology

Lanzhou University

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What is a mixed-effects model?

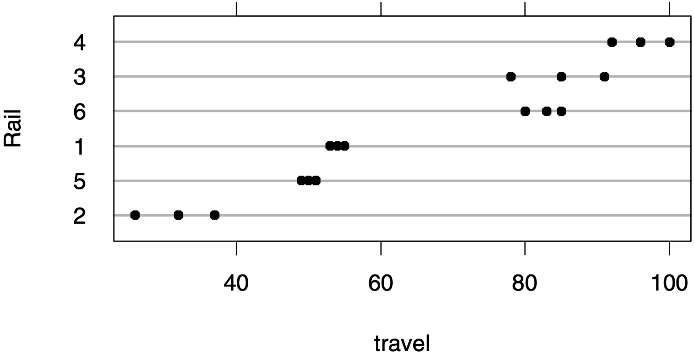
Fixed effects are parameters associated with an entire population or with certain repeatable levels of experimental factors.

Random effects are parameters associated with individual experimental units drawn at random from a population.

A linear model with both fixed effects and random effects is called a mixed-effects model.

A motivating example

Example: An experiment measured the time it took for certain type of ultrasonic wave to travel the length of rail. They chose six rails at random and tested each rail three times. They are interested in estimating the average time for a “typical” rail, the variation among rails and within a rail.



A motivating example

We may fit a linear model with rail identity as a categorical predictor. That is

$$y_{ij} = \mu + \beta_i + \varepsilon_{ij}$$

where i indexes the rail identity and j indexes the replicates for each rail. But this approach has several drawbacks:

- It does not account for correlation within a rail;
- It does not estimate the variation among rails and within a rail;
- Number of parameters increase linearly if we test more rails.

A motivating example

To address these shortcomings, we replace the fixed rail effect β_i with a random effect b_i. The model becomes

$$y_{ij} = \mu + b_i + \varepsilon_{ij}$$

where both b_i and ε_{ij} are assumed to follow normal distributions with mean 0 and unknown variances. This model has several advantages:

- Now the interpretation of μ changes from the mean over the 6 rails in this experiment to the mean over the population of all rails from which the 6 rails were sampled;
- We can estimate the rail to rail variability σ_b^2;
- The number of parameters no longer increases with the number of rails tested in the experiment.

Formulating mixed-effect models

In general, a linear mixed effects model express the response as the sum of fixed effects, random effects, and random errors. It can be generally written as

$$y = \beta_0 + \beta_1 X_1 + \ldots + b_0 + b_1 X_1 + \ldots + \varepsilon$$

Or more briefly in matrix form as

$$\boldsymbol{Y} = \boldsymbol{X}\beta + \boldsymbol{Zb} + \varepsilon$$

Here, both $\boldsymbol{b}$ and ε are assumed to follow normal distributions with mean of 0.

Mixed-effect models

A few comments on fixed and random effects:

- Random effects in principle means that the it is a random sample from the population of interest. In practice, whether we treat a variable as fixed or random effects primarily depends on the scope of inferences.
- Although random errors are typically assumed to follow iid normal distributions, we do not necessarily make such an assumption for random effects. In fact, random effects can assume complex patterns of variance-covariance structures.
- In mixed effects models, the parameters we estimates are the values of fixed effect parameters and the variance of the random effects. The values for the random effects can be predicted but it is not regarded as a parameter in the models.

Fitting mixed-effect model

Fitting mixed-effect model is generally done by the method of maximum likelihood. It typically consists of the following step:

- Treat parameter associated with random effects as known and maximize the likelihood function with respect to the fixed effects;
- Plug in the estimator of fixed effects back into the likelihood, which yield a likelihood function only as a function of parameters associated with the random effects. This is called profile likelihood. Maximize this function to obtain the estimates for random effects parameters;
- Use the estimates of random effects parameters to obtain parameter estimates for all fixed effects.

Restricted maximum likelihood estimates

Maximum likelihood estimation of variances produced biased estimators. This primarily arises from the fact that maximum likelihood estimators ignore the fact that parameter in the mean have been estimated.

Restricted maximum likelihood (REML) remedies this issues by applying the likelihood method to error contrasts instead of the original data.

- REML is a method of estimating the variance parameters, not all parameters in the model;
- REML cannot be used for likelihood ratio test. For example, when using AIC to compare model fits, model should not be fit by REML.

Inferences in linear mixed-effects model

Because mixed models are fit using maximum likelihood, standard likelihood-based inference techniques for fixed effects are available, such as Wald test or likelihood ratio test.

However, these likelihood based tests relies on asymptotic properties of maximum likelihood estimates. Within small sample size, these tests are known to be anti-conservative, i.e., they produce too small p-values and give false significance.

In linear mixed model, inferences are usually done using:

- parametric bootstrapping based inference;
- F-tests with degrees of freedom adjustments.

Inferences in linear mixed-effects model

While it is rarely of interest to perform hypothesis tests on the random effects, it may be a goal of data analysis in specific cases.

Likelihood based inference, such as likelihood ratio test, can be used for testing random effects. But note that these tests are only approximately correct when testing if the variance of a random effect is zero. This is because variance can only be positive and we are testing the parameter near its boundary of possible range.

Choosing proper model structure

How do I choose a proper model structure?

- Specify a saturated, or “full” model for the fixed effect structure, and then try to identify a parsimonious but adequate variance-covariance structure through the specification of random effects using AIC or other model comparison metric. Here, model should be fit with ML.
- If reduction of the fixed effect structure is desired, using F-tests with degrees of freedom adjustment, rather than likelihood based method.
- Refit the model using REML and use it as the basis for all final inferences.

Choosing proper model structure

A few comments on choosing fixed effects in the model:

- Using F-tests to select fixed effects runs the risk of type II error. That is, if we remove a fixed effect when it is not significant, we essentially accept the null hypothesis.
- If we analyze data from a designed experiment, the fixed effect should reflect the purpose of that design. In this case, we should not do model selections on fixed effects.
- If our goal is to build a parsimonious model for prediction, then using F-tests for model selection is appropriate.